Chapter 2: Probability Distribution

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Probability distribution: It is called expected frequency distribution in which frequencies are mathematically deduced under certain assumptions. They are of two types

Discrete probability distribution: The variable is discrete and takes whole number (integral) values. e.g. Binomial distribution, Poisson distribution

Continuous probability distribution: The variable is continuous and takes fractional values also. e.g. Normal distribution

i. **Binomial distribution:** The probability mass function is given by

$P(x) = {}^nC_x p^x q^{n-x}$ where n and p are called parameters

Criteria of binomial distribution:

- ✓ The number of observations or trials is fixed.
- ✓ Each observation or trial is independent.
- ✓ The probability of success is the same from trial to trial.
- ✓ There are only two possible outcomes in a trial.

ii. **Poisson distribution:** It gives the probability of a given number of events happens in a fixed interval of time. The probability mass function is given by

$P(x) = \frac{e^{-\lambda}.\lambda^x}{x!}$ where λ be the parameter and mean = variance = λ.

Criteria of Poisson distribution:

- ✓ The value of p is very small i. e. $P \to 0$
- ✓ The value of n is very large, i. e. $n \to \infty$

iii. Normal distribution: It is a continuous distribution and also called Gaussian distribution. The normal curve is the bell-shaped and symmetric curve. The probability density function is given by

$$f(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2}$$

Where, μ = mean and σ = standard deviation.

Standard normal distribution: If μ = 0 and σ = 1 the normal distribution is called standard normal distribution and $z = \frac{x-\mu}{\sigma}$ is called standard normal variate.

…ility of failure

Properties of Binomial Distribution

Binomial distribution has the following properties:

i) Discrete probability distribution takes only the positive integers values like 0, 1, 2, ...

ii) It has two parameters n and p (i.e. bi-parametric).

iii) The shape of the distribution depends on the value of n and p.

If $p = q = \frac{1}{2}$, the distribution is symmetric

$p > 0.5$, the distribution is negatively skewed

$p < 0.5$, the distribution is positively skewed

iv) Mean = np and $\sigma^2 = npq$ i. e. mean > variance

v) Binomial distribution tends to normal distribution if $p = q = 0.5$

Example 3

Test for impurities commonly found in drinking water from private wells showed that 30% of wells in a particular have impurities A. If a random sample of five wells is selected from a large number of wells in the country, what is the probability that

a. Exactly 3 will have impurities A.

b. At least 3 will have impurities A.

c. At most 3 will have impurities A.

Solution:

$X \sim b(n, p)$ then, the probability mass function is given by,

$$p(x) = {}^nC_x\, p^x q^{n-x}$$

If n = 5 and p = 30% = 0.3, then q= 1 – p = 1- 0.3= 0.7

a. $P(X=3) = {}^5C_3\, p^3 q^{5-3} = 10\,(0.3)^3(0.7)^2 = 0.1323$

b. $P(x \geq 3) = P(X = 3) + P(X = 4) + P(X = 5)$

$= {}^5C_3\, p^3 q^{5-3} + {}^5C_4\, p^4 q^{5-4} + {}^5C_5\, p^5 q^{5-5}$

$= 10\,(0.3)^3(0.7)^2 + 5\,(0.3)^4(0.7)^1 + 1\,(0.3)^5(0.7)$

$= 0.1323 + 0.0283 + 0.00243$

$= 0.16303$

c. $P(X \leq 3) = 1 - P(X > 3) = 1 - \{p(4) + p(5)\}$

$= 1 - \{0.0283 + 0.00243\}$

$= 0.9692$

Example 4

The mean and variance of a binomial distribution are 4 and $\frac{4}{3}$ respectively. Find the value of $P(X \geq 1)$

Solution:

Let $X \sim b(n, p)$, then we have,

Mean = E(X) = n p = 4

Variance = V(X) = n p q = $\frac{4}{3}$

$$\therefore \frac{npq}{np} = \frac{4}{3} \times \frac{1}{4}$$

Hence, $q = \frac{1}{3}$

and $p = 1 - q = 1 - \frac{1}{3} = \frac{2}{3}$

$\therefore n = \frac{4}{p} = 4 \times \frac{3}{2} = 6$

Again, $P(X \geq 1) = 1 - P(X = 0)$

$= 1 - {}^{6}C_{0}\left(\frac{2}{3}\right)^{0}\left(\frac{1}{3}\right)^{6-0}$

$= 1 - \left(\frac{1}{3}\right)^{6} = 0.998$

Example 5

A multiple-choice test consists of 10 questions and 4 answers to each question. If each question is answered by suffering 4 tags tabled 1,2,3 and 4 drawing one and making the alternative whose number of drawn. Find the probability of

a. Getting 3
b. At least 9
c. At most 2

Solution:

Here, the probability of selecting a correct answer from four alternatives, $p = \frac{1}{4} = 0.25$

and $q = 1 - p = 1 - 0.25 = 0.75$

$X \sim b\,(10, 0.25)$, then

a) $P(X = 3) = {}^{10}C_{3}\,(0.25)^{3}\,(1 - 0.25)^{10-3}$

$= 120 \times 0.01562 \times 0.13348$

$= 0.2502$

b) $P(X \geq 9) = P(X = 9) + P(X = 10)$

$= {}^{10}C_{9}(0.25)^{9}(1 - 0.25)^{10-9} + {}^{10}C_{10}(0.25)^{10}(1 - 0.25)^{10-10}$

$= 10 \times 3.8146 \times 10^{-6} \times 0.75 + 1 \times 9.536 \times 10^{-7} \times 1$

$= 2.861 \times 10^{-5} + 9.536 \times 10^{-7}$

$= 2.9563 \times 10^{-5}$

=0.00002956

c) P (X ≤ 2) = P (X=0) + P (X = 1) + P(x = 2)

$= {}^{10}C_0(0.25)^0(1-0.25)^{10-0} + {}^{10}C_1(0.25)^1(1-0.25)^{10-1} + {}^{10}C_2(0.25)^2(0.75)^8$

= 0.014 + 0.1877 + 0.2815 = 0.4832

Example 8

Fit a Poisson distribution to the following data.

No of Defects	4	3	2	1	0
Frequency	2	3	19	65	11

Solution:

Let x denote the number of defectives.

Here, N = Σf = 100

Mean, $\lambda = \frac{1}{n}\sum f\,x = \frac{120}{100} = 1.2$

and expected frequency f(x) = N p(x), x = 0, 1, 2, 3, 4.

No. of Defects	$P(x) = \frac{e^{-\lambda}\lambda^x}{x!}$	Expected Frequency
0	$P(0) = e^{-1.2} = 0.30$	100. P(0) = 30
1	$P(1) = \frac{1.2}{0+1}P(0) = 0.36$	100. P(1) = 36
2	$P(2) = \frac{1.2}{1+1}P(1) = 0.21$	100. P(2) =22
3	$P(3) = \frac{1.2}{2+1}P(2) = 0.08$	100. P (3) =9
4	$P(4) = \frac{1.2}{3+1}P(3) = 0.02$	100. P (4) =3

Standard Normal Distribution

A normal distribution of continuous random variable Z with mean, (μ)= 0 and variance (σ^2) = 1 is called standard normal distribution and Z is called standard normal variate. The p. d. f. of the standard normal distribution is given by

$$\phi(z) = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}(z)^2}, -\infty < Z < \infty$$

i.e. if X~ N (μ, σ^2) then, $z = \frac{x-\mu}{\sigma}$ is a standard normal variate with E(Z) = 0 and var (Z) = 1

Mathematically, it is written as Z~N(0,1) and the probability distribution function of Z is given as

$$\phi(z) = P(Z < z) = \int f(z)dz = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{z} e^{\frac{-z^2}{2}} dz$$

Example 11

The average height of students in a class is 165cm and the standard deviation is 10cm. Find the percentage of students whose height lies between 150 cm and 180cm.

Solution:

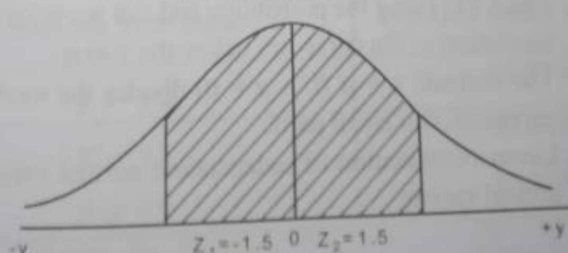
Here,

X~ N (μ, σ^2) then, $z = \frac{X-\mu}{\sigma} \sim N(0,1)$

Mean, μ= 165 cm and s. d., σ = 10 cm

$$\therefore P(150 \le X \le 180) = P\left(\frac{150-165}{10} \le \frac{X-\mu}{\sigma} \le \frac{180-165}{10}\right)$$

$$= P(-1.5 \le Z \le 1.5)$$



$$= F(1.5) + F(-1.5)$$

$$= P(-1.5 \text{ to } 0) + P(0 \text{ to } 1.5)$$

$$= 0.4332 + 0.4332$$

$$= 0.8664 \times 100 = 86.64\%$$

Example 1

Fit a binomial distribution of the following data

X	0	1	2	3	4	5	6
F	5	18	28	12	7	6	4

Also calculate the mean, standard deviation and variance of binomial distribution from the data.

Solution:

Calculation of mean

X	F	f x
0	5	0
1	18	18
2	28	56
3	12	36
4	7	28
5	6	30
	$N = \sum f = 80$	$\Sigma fx = 192$

$$\bar{x} = \frac{\sum fx}{N} = \frac{192}{80} = 2.4$$

Let, X ~ B (n, p)

Where, n = number of trials = 6

p = probability of success

q = 1 - p = probability of failure.

For binomial distribution,

$$\bar{x} = np = 2.40$$

$$P = \frac{\bar{x}}{n} = \frac{2.4}{6} = 0 \text{ and } q = 1 - p = 0.60$$

Example 3

A music club in Kathmandu has 1000 members and probability that a member demands the record of a particular singer is 0.15, at least how much number of records should club keep in order to ascertain that the chance of accepting request for the record of any singer is more than 0.90. (2066

Solution:

At least any singer is more than 0.90 means that.

Given, P = 0.15 and q = 0.85 then find n =?

$P(X \geq 1) = 0.90$

$1 - P(X = 0) = 090$

$1 - {}^{n}c_0\ (0.15)^0 * (0.85)^{n-0} = 0.90$

$1-(0.85)^n = 0.90$ further taking -log both sides.

$-(0.85)^n = -(0.10)$ further taking -log both sides.

$n * \log 0.85 = \log 0.10$ by simplification

$n = 14$ (Approx).

Example 4

A binomial random variable X satisfies the relation 6*p(X=3) = P(X = 2), when n = 5. Find the values of the parameter " P". (2015 TU Semester).

Solution :

The p.m.f. of binomial distribution.

$P(X = r) = {}^{n}c_r\ p^r\ q^{n-r}$

$P(X = 3) = {}^{5}c_3\ p^3\ q^{5-3}$

$P(X = 2) = {}^{5}c_2\ p^2\ q^{5-2}$

By the given condition

$6*p(X=3) = P(X = 2),$

$6 * {}^{5}c_3\ p^3\ q^{5-3} = {}^{5}c_2\ p^2\ q^{5-2}$ where $({}^{5}c_3 = {}^{5}c_2)$

$6*p = q$ where$(p + q = 1)$

$6*p = 1-p$

$7*p = 1$

$P = \frac{1}{7} = 0.14$

The expected frequency is

$f(x) = N.P(X = x) \quad = 80\ {}^{n}C_{x}\, p^{x} q^{n-x}$

$= 80\ {}^{6}C_{x}\,(0.4) \times (0.60)^{6-x},\ x = 0, 1, 2, \ldots 6$

Now,

$f(0) = 80.{}^{6}C_{0}\,(0.4)^{0}.\,(0.6)^{n-0} = 3.73 \approx 4$

$f(1) = 80.{}^{6}C_{1}\,(0.4)^{1}.\,(0.6)^{n-1} = 14.93 \approx 15$

$f(2) = 80.{}^{6}C_{2}\,(0.4)^{2}.\,(0.6)^{n-2} = 24.88 \approx 25$

$f(3) = 80.{}^{6}C_{3}\,(0.4)^{3}.\,(0.6)^{n-3} = 22.12 \approx 22$

$f(4) = 80.{}^{6}C_{4}\,(0.4)^{4}.\,(0.6)^{n-4} = 10.06 \approx 10$

$f(5) = 80.{}^{6}C_{5}\,(0.4)^{5}.\,(0.6)^{n-5} = 2.95 \approx 3$

$f(6) = 80.{}^{6}C_{6}\,(0.4)^{6}.\,(0.6)^{n-6} = 0.33 \approx 1$

The fitted binomial distribution is given as

X	0	1	2	3	4	5	6	Total
Observed frequency	5	18	28	12	7	6	4	**80**
Expected frequency	4	15	25	22	10	3	1	**80**

Now,

Mean = np =6 x 0.4 =2.4

And variance = n p q = 80 x 0.4 x 0.6 =1.44

Standard deviation = $npq = \sqrt{1.44} = 1.2$

Example 2

Fit a binomial distribution to the following data:

X	0	1	2	3	4
F	4	10	46	62	28

Solution:

Calculation of mean

X	f	Fx
0	4	0
1	10	10
2	46	92
3	62	186
4	28	112
Total	$N=\sum f=150$	$\Sigma f x = 400$

$\bar{x} = \frac{\sum fx}{N} = \frac{400}{150} = 2.67$

Let, $X \sim B(n, p)$

Where, n = number of trials = 4

p = probability of success

q = 1 - p = probability of failure.

For binomial distribution,

$\bar{x}$= np = 2.67

$\therefore P = \frac{\bar{x}}{n} = \frac{2.67}{4} = 0$ and q = 1 - p = 0.33

The expected frequency is

$f(x) = N.P(X = x) \quad = 400\ {}^{n}C_{x}\, p^{x} q^{n-x}$

$= 150\ {}^{4}C_{x}\,(0.67) \times (0.33)^{6-x}$, x = 0, 1, 2, ... 4

Now,

$f(0) = 150\ .{}^{4}C_{0}(0.67)^{0}.\ (0.33)^{4-0} = 1.78 \sim 2$

$f(1) = 150\ .{}^{4}C_{1}\,(0.67)^{1}.\ (0.33)_{4-1} = 14.45 \sim 14$

$f(2) = 150\ .{}^{4}C_{2}\,(0.67)^{2}.\ (0.33)_{4-2} = 43.99 \sim 44$

$f(3) = 150\ .{}^{4}C_{3}\,(0.67)^{3}.\ (0.33)_{4-3} = 59.55 \sim 60$

$f(4) = 150\ .{}^{4}C_{4}\,(0.67)^{4}.\ (0.33)_{4-4} = 30.23 \sim 30$

The fitted binomial distribution is given as

X	0	1	2	3	4	Total
Observed frequency	4	10	46	62	28	150
Expected frequency	2	14	44	60	30	150

Example 5

Fit a Poisson distribution for the following frequencies distribution

Number of defects	4	3	2	1	0
Frequency	2	3	19	65	111

Solution:

Calculation of mean

Number of defects (X)	Frequency (f)	f X
4	2	8
3	3	9
2	19	38
1	65	65
0	111	0
Total	$N=\Sigma f=200$	$\Sigma fx = 120$

$$\bar{x} = \frac{\Sigma fx}{N} = \frac{120}{200} = 0.6$$

Here, $X \sim P(\lambda)$, λ = mean = 0.6

Then

$$P(X=x) = \frac{e^{-\lambda}.\lambda^{x}}{x!}, x = 0, 1, 2, 3, 4$$

The expected frequency is calculated by

$$f(x) = N.\ p(X=x) = \frac{e^{-\lambda}.\lambda^{x}}{x!} = \frac{e^{-0.6}.(0.6)^{x}}{x!}, x = 0, 1, 2, 3, 4$$

Then, $f(0) = 200.\ P(0) = 200.\frac{e^{-0.6}.(0.6)^{x}}{0!} = 109.76 = 110$

$$f(1) = 200.\ P(1) = 200.\frac{e^{-0.6}.(0.6)^{x}}{1!} = 65.86$$

$$f(2) = 200.\ P(2) = 200.\frac{e^{-0.6}.(0.6)^{x}}{2!} = 19.75 = 20$$

$$f(3) = 200.\ P(3) = 200.\frac{e^{-0.6}.(0.6)^{x}}{3!} = 3.95 = 4$$

$f(4) = 200.\ P(4) = 200.\ \frac{e^{-0.6}.(0.6)^x}{4!} = 0.59 = 0$

X	0	1	2	3	4	Total
Observed frequency	111	65	19	3	2	200
Expected frequency	110	66	20	4	0	200

Example 6:

If a random variable X follows poisson distribution such that P(X = 1) = P(X = 2), find

(i) The mean and variance of the distribution.

(ii) Find P(X = 0)

Solution:

The P.m.f of poisson distribution

$P(X = x) = \frac{e^{-\lambda}\lambda^x}{x!}$ Where, x = 0 , 1 ,2 , 3 …….. (A*)

Putting x = 1 and 2 in equation (A*) , we get following results

$$P(X = 1) = \frac{e^{-\lambda}\lambda^1}{1!}$$

$$P(X = 2) = \frac{e^{-\lambda}\lambda^2}{2!}$$

We are given that :

P(X = 1) = P (X = 2)

$\frac{e^{-\lambda}\lambda^1}{1!} = \frac{e^{-\lambda}\lambda^2}{2!}$

$\lambda = 2$ In poisson distribution mean = variance = 2

(ii) P(X = 0) Putting x = 0 and $\lambda = 2$ in (A*) we get

$$P(X = 0) = \frac{e^{-2}\lambda^0}{0!}$$

$$= 0.1353$$

Example 7:

Incomes of a group of 10,000 persons were found to be normally distributed with a mean of Rs. 52,000 and standard deviation of Rs. 6,000. Find (I) the number of persons having income between Rs. 40,000 and Rs. 55,000 (ii) the lowest income of the richest 1,000 persons (iii) the highest income of the poorest 1,000 persons (iv) the Lowest income of the richest 20% of the persons.

Solution:

Here, the Number of persons, N = 10,000

Mean income $(\mu) = Rs. 5200$

Standard deviations, = Rs. 6,000

Let, the random variable X denotes the income of the persons such that $X \sim N(\mu, \sigma^2)$.

The corresponding standard normal variate is

$Z = \frac{x-\mu}{\sigma} = \frac{x-52000}{6000} \sim N(0, 1)$.

The probability that the persons having income between Rs. 40,000 and Rs. 55,000

$= P(40{,}000 < X < 56{,}000)$.

$= P\left(\frac{40000-52000}{6000} < \frac{x-52000}{6000} < \frac{55000-52000}{6000}\right)$

$= P(2 < Z < 0) + P(0 < Z < 0.5)$

$= P(0 < Z < 2) + P(0 < Z < 0.5)$, by symmetry

The number of persons having income between Rs. 40,000 and Rs. 55,000

$= N.P(40{,}000 < X < 55{,}000)$

$= 10{,}000 \times 0.6687$

$= 6687$

ii. Let, the lowest income of richest 1,000 persons e.$\frac{1000}{10000} \times 100 = 10\%$) be x_1, then

$P(X > x_1) = 0.10$

or, $P\left(\frac{x-52000}{6000} > \frac{x_1-52000}{6000}\right) = 0.10$

or, $P(Z > Z_1) = 0.10$

where $Z_1 = \frac{x_1-52000}{6000}$ .... (i)

$\therefore$ $P(0 < Z < z_1) = 0.4$

From the normal table, $P(0 < Z < 1.28) = 0.3997 = 0.4$

$\therefore$ $Z1 = 1.28$

From equation (i), we have $1.28 = \frac{x_1-52000}{6000}$

$\therefore$ $x_1 = 5200 + 1.28 \times 6000$

or, $x_1 = 59680$

Hence, the lowest income of the richest 1000 persons is 59680.

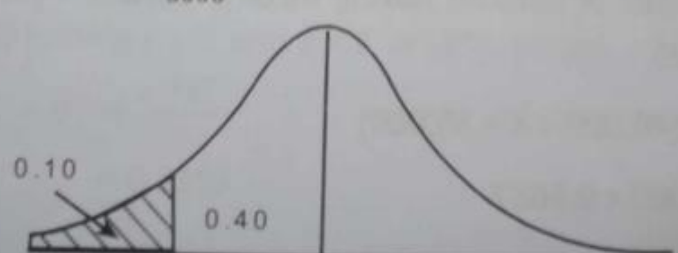
iii. Let, the highest income of poorest 1,000 persons

(i. e.$\frac{1000}{10000} \times 100 = 10\%$) be x_2, then $P(X > x_2) = 0.10$

or, $P\left(\frac{x-52000}{6000} < \frac{x_2-52000}{6000}\right) = 0.10$

or, $P(Z < Z_2) = 0.10$

where $Z_2 = \frac{x_2-52000}{6000}$ .... (ii)



$\therefore P(-Z_2 < Z < 0) = 0.04$

or, $P(0 < Z < Z_2) = 0.40$

From the normal table, $P(0 < Z < 1.28) = 0.3997 = 0.04$

$\therefore Z_2 = 1.28$

From equation (i), we have, $-1.28 = \frac{x_2 - 52000}{6000}$

$\therefore x_2 = 5200 - 1.28 \times 6000$

or, $x_2 = 44320$

Hence, the highest income of the poorest 1000 persons is 44320.

iv. Let, the lowest income of richest 20% be x_3, then

$P(X > x_1) = 0.10$

or, $P\left(\frac{x - 52000}{6000} > \frac{x_3 - 52000}{6000}\right) = 0.20$

or, $P(Z > Z_3) = 0.20$

where $Z_3 = \frac{x_3 - 52000}{6000}$ ....(iii)

$\therefore P(0 < Z < z_3) = 0.3$

From the normal table, $P(0 < Z < 1.28) = 0.3997 = 0.04$

$\therefore Z_3 = 0.84$

From equation (iii), we have $0.84 = \frac{x_3 - 52000}{6000}$

$\therefore x_3 = 5200 + 0.84 \times 6000$

or, $x_1 = 57040$

Hence, the lowest income of the richest 20% of the persons is 57040.

Example 8

In an examination, 10% of the students got less than 30 marks and 95% of the students' got less than 85 marks. Assuming the distribution to be normal, find the mean and standard deviation of the mark's distribution.

Solution:

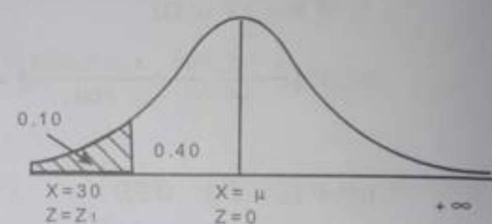
Let, the random variable X denotes the income of the persons such that $X \sim N(\mu, \sigma^2)$.

The corresponding standard normal variate is $Z = \frac{X-\mu}{\sigma} \sim N(0, 1)$.

Given,10% of the students get less than 30 marks

i. e., $P(X < 30) = 0.10$

or, $P\left(\frac{x-\mu}{\sigma} < \frac{30-\mu}{\sigma}\right) = 0.10$



or, $P(Z < -z_1) = 0.1$

Where, $\frac{30-\mu}{\sigma} = -z_1$ .....(i)

or, $0.5 - P(-z_1 < Z < 0) = 0.1$

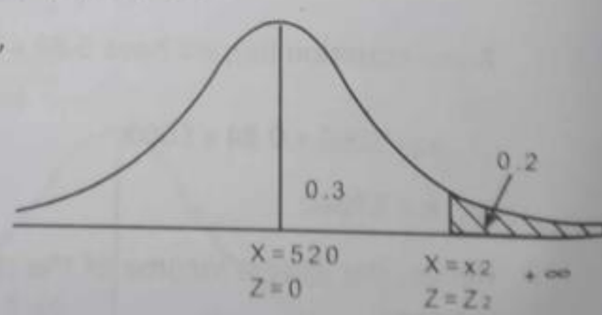
or, $P(-z_1 < Z < 0) = 0.4$

or, $P(0 < Z < z_1) = 0.4$, by symmetric

$P(0 < Z < 1.28) = 0.4$,

$\therefore Z_1 = 1.28$

From equation (i)



$\frac{30-\mu}{\sigma} = -1.28$

or, $30 - \mu = -1.28$

or, $\mu = 30 + 1.28\sigma$ .... (ii)

Again, 95% of the students got less than 85 marks

$P\left(\frac{x-\mu}{\sigma} < \frac{85-\mu}{\sigma}\right) = 0.95$

or, $P(Z < Z_2) = 0.95$

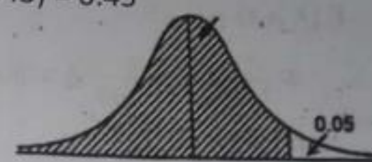
Where, $\frac{85-\mu}{\sigma} = Z_2$ .... (iii)

or, $0.5 + P(0 < Z < Z_2) = 0.95$

or, $P(0 < Z < Z_2) = 0.45$

But from normal table $P(0 < Z < 1.645) = 0.45$

$\therefore z_2 = 1.645$



From equation (iii), $\frac{85-\mu}{\sigma} = 1.645$

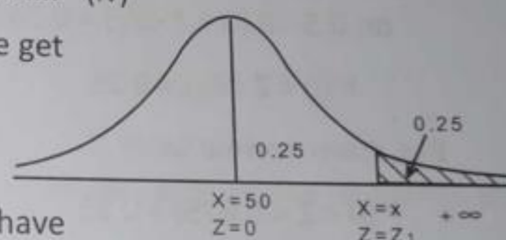
or, $85 - \mu = 1.645\sigma$

$\therefore \mu = 85 - 1.645\sigma$ .... (iv)

From equation (ii) and (iv), we get

$30 + 1.28\sigma = 85 - 1.645\sigma$

$\therefore \sigma = 18.80$



Again, from equation (ii), we have

$\mu = 30 + 1.28 \times 18.80 = 54.064$

So, the mean and standard deviation of the mark's distribution are 54.064 and 18.80 respectively.

Example 9

The marks obtained by 3000 students in an examination are normally distributed with a mean 60 and a standard deviation of 10.

i. Find the lowest marks of the top 25% of the students.

ii. Find the limits of the middle 60% of the students.

Solution:

Let the random variable X denotes the marks obtained by 300 students such that

$X \sim N(\mu, \sigma^2)$

Where, mean (μ) = 60 and Standard deviation (σ) = 10

The corresponding standard normal variate is

$$Z = \frac{x-\mu}{\sigma} = \frac{x-60}{10} \sim N(0, 1)$$

i. Let the lowest marks of top 25% of the students, be x, then

$P(X > x) = 0.25$

$$P\left(\frac{X-60}{10} > \frac{x-60}{10}\right) = 0.25$$

$P(Z > Z_1) = 0.25$

Where, $Z_1 = \frac{x-60}{10}$ ... (i)

or, $0.5 - P(0 < Z < Z_1) = 0.25$

$\therefore P(0 < Z < Z_1) = 0.25$

But, from normal table

$P(0 < Z < 0.675) = 0.25$

$\therefore Z_1 = 0.675$

From equation (i)

$$\frac{x-60}{10} = 0.675$$

or, $x = 60 + 0.675 \times 10 = 66.7$

Hence the lowest mark of the top 25% of the students is 66.75 mark.

ii. Let x_1 and x_2 be the limits of marks of middle 60% of the students, then

$P(x_1 < X < x_2) = 0.6$

$$P\left(\frac{X_1-60}{10} < \frac{X-60}{10} < \frac{X_2-60}{10}\right) = 0.60$$

$P(-Z_1 < Z < Z_2) = 0.60$

Where, $\frac{X_1 - 60}{10} = -Z_1$ ...(ii)

And $\frac{X_2 - 60}{10} = Z_2$ ...(iii)

or, $2P(0 < Z < Z_1) = 0.60$

or, $P(0 < Z < Z_1) = 0.30$

From the normal table,

$P(0 < Z < 0.84) = 0.2995 = 0.3$

$Z_1 = 0.84$

From equation (ii)

$$\frac{X_1 - 60}{10} = -0.84$$

$x_1 = 60 - 0.84 \times 10 = 51.60$

From equation (iii)

$$\frac{X_1 - 60}{10} = 0.84$$

$x_2 = 60 + 0.84 \times 10 = 68.40$

Hence, the limits of marks of middle 60% of the students are 51.60 and 68.40.

Example 10

The marks obtained by 5000 students in an examination are normally distributed with. mean 65 and standard 'deviation 20. Estimate

(I) lowest marks of top 10% students

(II) (ii) highest marks of poorest 50students.

Solution:

Here, the number of students, N = 5000

Mean marks, $\mu = 65$

Standard deviation, $\sigma = 20$

Let, the random variable X denotes the marks obtained by the students and $X \sim N(\mu, \sigma^2)$

The corresponding standard normal variable is

$$z = \frac{x-\mu}{\sigma} = \frac{x-65}{20} \sim N(0.1)$$

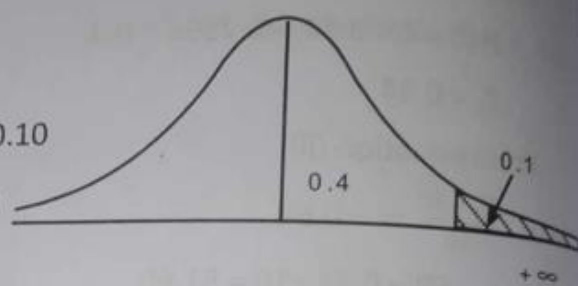
(i) Let the lowest marks of top 10% students be x_1, then

$P(X > x_1) = 10\%$

$P(X > x_1) = 0.10$

$P\left(\frac{X-\mu}{\sigma} > \frac{X-65}{20}\right) = 0.10$

or, $P(Z > Z_1) = 0.10$



then, $Z_1 = \frac{X_1 - 65}{20}$

or, $P(0 < Z < Z_1) = 0.4$

From the normal table

$P(0 < Z < 1.28) = 0.4$

$\therefore Z_1 = 1.28$

From the equation (i)

$\frac{X_1 - 65}{20} = 1.28$

or, $x_1 - 65 = 1.28 \times 20$

$X_1 = 65 + 1.28 \times 20 = 90.6$

Hence, the lowest mark of the top 10% of students is the 90.6 mark.

ii. Let the highest marks of poorest 50 students (i.e. $\frac{50}{5000} \times 100 = 10\%$) be X_2 then

$P(X < x_2) = 10\%$

$P(X < x_2) = 0.1$

$P\left(\frac{X-65}{20} < \frac{X_2-65}{20}\right) = 0.1$

or, $P(Z < -Z_2) = 0.1$

Where, $-Z_2 = \frac{X_2-65}{20}$ ...(ii)

or, $P(-Z2 < Z < 0) = 0.4$

or, $P(0 < Z < Z_2) = 0.4$ (by symmetry)

From the normal table,

$Z_2 = 1.28$

From equation (ii)

$-1.28 = \frac{X_2-65}{20}$

or, $x_2 - 65 = 1.28 \times 20$

or, $x_2 = 65 - 1.28 \times 20$

$x_2 = 39.4$

Hence, the highest mark of the poorest 500 students is 39.

Example 11

Out of 800 families of Kathmandu with 4 children each, what percentage would be expected to have (I) 2 boys and 2 girls (ii) at least one boy (iii) no girls and (iv) at most 2 girls. Assume equal probabilities for boys and girls.

Solution:

Here;

Number of families, N=800

Number of children, n = 4

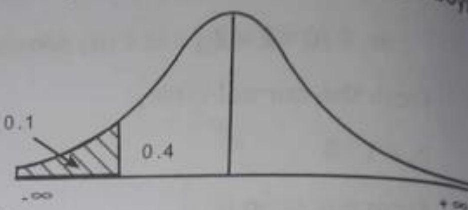
Probability of boy = Probability of girl $= \frac{1}{2}$

Let the random variable X denotes the number of boys such that $X \sim B(n, p)$

Where, n = 4, p = probability of boy = $\frac{1}{2}$

$q = 1 - p = \frac{1}{2}$

From the binomial distribution, the probability of exactly x boys is



$P(X = x) = {}^{n}c_x p^x q^{n-x}$

$= {}^{4}C_x \left(\frac{1}{2}\right)^x \cdot \left(\frac{1}{2}\right)^{n-x}$

$= {}^{4}C_x \left(\frac{1}{2}\right)^4, x = 0, 1, 2, 3, 4$

i. The probability that the families have 2 boys and 2 girls is

$P(X = 2) = \frac{1}{16} {}^{4}C_2 = 0.375 = 37.5\%$

37.5% of the families have 2 boys and 2 girls.

ii. The probability that the families have at least one boy is

$P(X \geq 1) = 1 - P(X = 0)$

$= 1 - \frac{1}{16} . {}^{4}C_2$

$= 1 - \frac{1}{16} = \frac{15}{16} = 0.9375 = 93.75\%$

93.75% of the families have at least one boy.

iii. The probability that the families have no girls i. e. all 4 are boys is

$P(X = 4) = \frac{1}{16} {}^{4}C_4 = \frac{1}{16} = 0.0625 = 6.25\%$

25% of the families have no girls.

iv. The probability that the families have at most 2 girls

$P(X \leq 2) = P(X = 0) + P(X = 1) + P(X = 2)$

$= {}^{4}C_0 (0.50)^0 (0.50)^4 + {}^{4}C_1 (0.50)^1 (0.50)^3 + {}^{4}C_2 (0.50)^2 (0.50)^2$

$= 0.0625 + 0.3125 + 0.375$

$= 0.75$

Example 12

A binomial random variable X satisfies the relation 6P (X = 3) =P(X = 2) when n = 5. Find the value of the parameter 'p'.

Solution:

Let X ~ B (n, p) where, n = 5, p =?

We have, $P(X = x) = {}^{n}c_{x} p^{x} q^{n-x}$, x = 0, 1, 2, 3, 4, 5

Given; 6.P(X = 3) = P (X = 2)

or, $6.\ {}^{5}c_{3} p^{3} q^{5-3} = {}^{5}c_{2} p^{2} q^{5-2}$

or, 6 x 10p = 10q

or, 6p = 1 - p (∵ q = 1 - p)

or, 7p = 1

or, $P = \frac{1}{7}$

Example 13

The number of industrial injuries per working week in a particular factory is known to follow a Poisson distribution with a mean of 0.5. Find the probability that in a particular week there will be: (i) less than 2 accidents, (ii) more than 2 accidents (iii) exactly 2 accidents.

Solution:

Let, the random variable X denotes the number of industrial injuries per working week in a factory such that X ~ P (λ),

Where,

Mean number of industrial injuries per working week = 0.5

From Poisson distribution, we have

$$P(X = x) = \frac{e^{-\lambda}\lambda^{x}}{x!} = \frac{e^{-0.5}(0.5)^{x}}{x!},\ x = 0, 1, 2, \ldots$$

i. The probability that in a particular week there will be less than 2 accidents is

$P(X<2) \quad = P(X \leq 1)$

$$= P(X=0) + P(X=1)$$

$$= \frac{e^{-0.5}(0.5)^0}{0!} + \frac{e^{-0.5}(0.5)^1}{1!}$$

$$= e^{-0.5}\left(\frac{(0.5)^0}{0!} + \frac{(0.5)^1}{1!}\right)$$

$P(X<2) \quad = 0.9098$

ii. The probability that in a particular week there will be more than 2 accidents

$$= P(X>2)$$

$$= 1 - P(X)$$

$$= 1 - [P(X=0) + P(X=1) + P(X=2)]$$

$$= 1 - [0.9098 + e^{-0.5}\frac{(0.5)^2}{2!}]$$

$$= 1 - (0.9098 + 0.0758)$$

$$= 1 - 0.9856$$

$P(X>2) \quad = 0.0144$

iii. The probability that in a particular week there will be exactly two accidents is

$$P(X=2) = e^{-0.5}\frac{(0.5)^2}{2!} = 0.0758$$

Example 14

Manufacture, who produce medicine bottles, find that 0.1% of the bottle are defective. The bottle is packed in boxes containing 500 bottles. Drug manufacture buys 100 boxes from the producer bottles. Using poison distribution find how many boxes will contain

(i) no defective (ii) one defective (iii) at least two defectives.

Solution:

Here, Probability of defective bottle, p = 0.1 % = 0.01

Number of bottles in a box, n = 500

Number of boxes, N = 100

Let, the random variable X denotes the number of defective bottles in the box such that $X \sim P(\lambda)$,

Where, λ = mean = np = 500 x 0.001

= 500 x 0.001

= 0.5

From Poisson distribution, we have,

$$P(X = x) = \frac{e^{-\lambda}\lambda^x}{x!} = \frac{e^{-0.5}(0.5)^x}{x!}, x = 0, 1, 2, \ldots$$

i. The number of boxes will contain no defective bottles is

$$N.P(X = 0) = 100 \times \frac{e^{-0.5}(0.5)^0}{0!} = 60.65 \approx 61$$

ii. The number of boxes will contain one defective bottle is

$$N.P.(X = 1) = 100 \times \frac{e^{-0.5}(0.5)^1}{1!} = 30.33 \approx 30$$

iii. The number of boxes will contain at least two defective bottles is

$$N.P.(X \geq 2) = N [1 - P(X = 0) - P(X = 1)]$$

$$= N [1 - P(X = 0) - N.P.(X = 1)]$$

$$= 100 \{1 - 0.61 - 0.30\}$$

$$= 100 \times 0.09 = 9$$

Example 15

If 3% of electric bulbs manufactured by a company are defective, find the probability that in a sample of 80 bulbs (i) exactly seven bulbs are defective (ii) at most two bulbs are defectives.

Solution:

Here, the Number of electric bulbs, n = 80

Probability of defective electric bulb, p = 3% = 0.03

Since, n = 80 is large and p = 0.03 < 0.05, Poisson approximation is a good approximation to

Binomial.

Let, the random variable X denotes the number of defective bulbs such $X \sim P(\lambda)$

Where, λ = mean = np = 80 x 0.03 = 2.4

$$P(X = x) = \frac{e^{-\lambda}\lambda^x}{x!} = \frac{e^{-2.4}(2.4)^x}{x!}, x = 0, 1, 2, \ldots$$

i. The probability that seven bulbs are defective is

$$P(X = 7) = \frac{e^{-2.4}(2.4)^7}{7!} = 0.0083$$

ii. The probability that at most two bulbs are defective is

$$P(X \leq 2) = P(X = 0) + P(X = 1) + P(X = 2)$$

$$= \frac{e^{-2.4}(2.4)^0}{0!} + \frac{e^{-2.4}(2.4)^1}{1!} + \frac{e^{-2.4}(2.4)^2}{2!}$$

$$= 0.5697$$

Example 16

The life of a certain product is assumed to be normally distributed with a mean of 160 hours and a standard deviation of 24 hours, what is the probability that the life of a randomly the chosen product is (i) less than 122 hours (ii) between 141 hours and 190 hours?

Solution:

Let the random variable. X denotes the life of a product such that X~N (μ, σ^2)

Where, μ= mean = 160 hours

σ= standard deviation = 24 hours

The corresponding standard normal variate is

$$Z = \frac{X-\mu}{\sigma} = \frac{X-160}{24} \sim N(0,1)$$

i. The probability that the life of a randomly chosen product is less than 122 hours is

$P(X < 122) = P\left(\frac{X-160}{24} < \frac{122-160}{24}\right)$

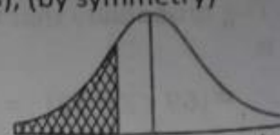
$= P(Z < -1.58)$

$= 0.5 - P(-1.58 < Z < 0)$

$= 0.5 - P(0 < Z < 1.58)$, (by symmetry)

$= 0.5 - 0.4429$

$P(X < 122) = 0.0571$

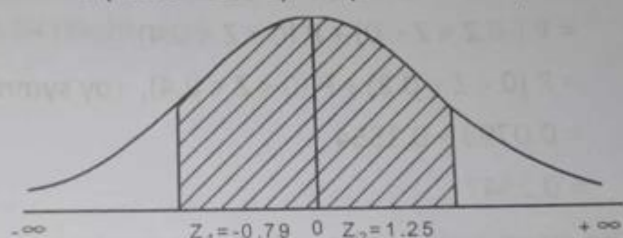


ii. The probability that the life. of a randomly chosen product is between 141 hours and 190 hours is

$P(141 < X < 190) = P\left(\frac{141-160}{24} < \frac{X-160}{24} < \frac{190-160}{24}\right)$

$= P(-0.79 < Z < 1.25)$

$= P(-0.79 < Z < 0) + P(0 < Z < 1.25)$



$= P(0<Z<0.79) + P(0<Z<1.25)$ (by symmetry)

$= 0.2852 + 0.3944$

$\therefore$ $P(141 < X < 190) = 0.6796$

Example 17

The weekly wages of 1000 workers are normally distributed with mean income of \$70 with p standard deviation of \$5. Estimate the number of workers whose weekly wages will be (i) between \$69 and \$72 (ii) less than \$63.

Solution:

Number of workers, N =1000

Mean income, μ = \$ 70

Standard deviation, σ = \$ 5

Let the random variable X denotes the weekly wages of wor
such that $X \sim N(\mu, \sigma^2)$

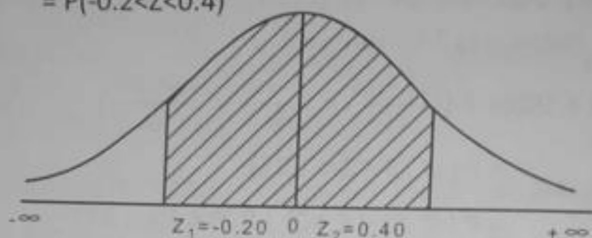
The corresponding standard normal variate is

$$Z = \frac{X-\mu}{\sigma} = \frac{X-160}{24} \sim N(0,1)$$

i. The probability that weekly wages will be between \$ 69 and \$ 7 is

$$P(69 < Z < 72) = P\left(\frac{69-70}{5} < \frac{X-70}{5} < \frac{72-70}{5}\right)$$

$= P(-0.2 < Z < 0.4)$



$= P(-0.2 < Z < 0) + P(0 < Z < 0.4)$

$= P(0 < Z < 0.2) + P(0 < Z < 0.4)$, (by symmetry)

$= 0.0793 + 0.1554$

$= 0.2347$

The expected number of workers whose weekly wages will be between \$69 and \$72 is

$N.P(69 < Z < 72) = 1000 \times 0.2347 = 234.7 \approx 235$

ii. The probability that weekly wages will be less than \$63 is

$P(X < 63) = P\left(\frac{X-70}{5} < \frac{63-70}{5}\right)$

$= P(Z < -1.4)$

$= 0.5 - P(0 < Z < 1.4)$

$= 0.5 - 0.4192$

$= 0.0808$

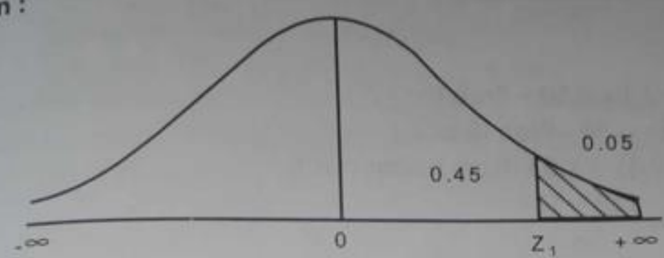
The expected number of workers whose weekly wages will be less than \$63 is

$N.P(X < 63) = 1000 \times 0.0808 = 80.80 \sim 81$

Example 18

Students of a class are given a test. Their marks were normally distributed with mean 73 and standard deviation is 8. Find only 5% of the students appearing in the exam. Find the score highest from which marks or mean marks?. (TU 2069).

Solution :



$P(Z > Z_1) = 0.50 - \text{prob } (0 \text{ to } z_1)$

$005 = 0.50 - \text{prob } (0\text{to } z_1)$

$Z_1 = 0.45 = (+)\ 1.64$ (Right side)

$$Z_1 = \frac{X - 73}{8}$$

$$1.64 = \frac{X - 73}{8}$$

$X = 86.16$

Hence, the minimum marks of top 5% of the students is 86.16 marks.

Example 19

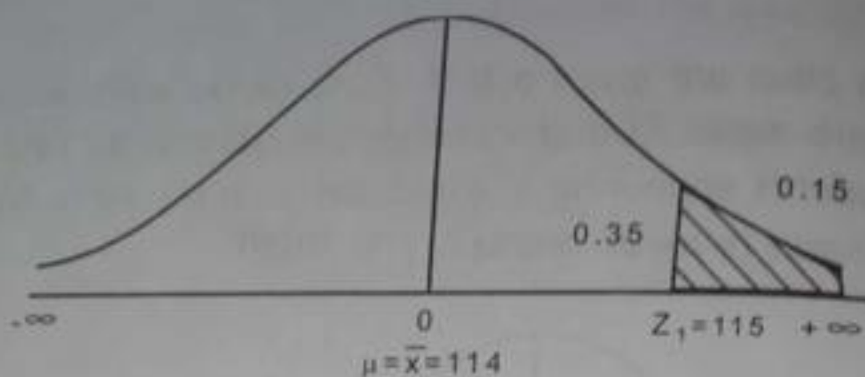
Sacks of grain packed by an automatic machine loader follows normal distribution having an average weights of 114 kgs. It is found that 15% of bags are over 115 kgs. Find the standard deviation (TU 2067).

Solution :

S.N.V. $Z = \frac{X - \mu}{\sigma} = \frac{X - 114}{\sigma}$

$P(X > X_1) = P(Z > Z_1) = 0.15$

Or $\quad = P(Z > 115) = 0.15$



$P(Z>Z_1) = 0.50 – \text{Prob } (0 \text{ to } Z_1)$

$0.15 = 0.50 - \text{Prob } (0 \text{ to } Z_1)$

$Z_1 = 0.35 = 1.03$ (from normal table).

$$1.03 = \frac{115 - 114}{\sigma}$$

Hence, the standard deviation is 0.971 kgs.